

PRODUCTS, HOMOTOPY LIMITS AND APPLICATIONS

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ABSTRACT. In this note, we discuss the derived functors of infinite products and homotopy limits. $QC(X)$, the category of quasi-coherent sheaves on a Deligne-Mumford stack X , usually has the property that the derived functors of product vanish after a finite stage. We use this fact to study the convergence of certain homotopy limits and apply it to compare the derived category of $QC(X)$ with certain other closely related triangulated categories.

CONTENTS

1. Introduction	1
2. Preliminaries	3
3. Products of Quasi-coherent sheaves on a Deligne-Mumford stack	5
4. Homotopy Limit of Truncations	6
5. Strong Serre subcategories	8
References	13

1. INTRODUCTION

Let \mathcal{A} be an abelian category with products and enough injectives and $D(\mathcal{A})$ be its unbounded derived category. One of the tools used to study the unbounded derived category is the concept of a homotopy limit as defined in [BN93]. If the abelian category \mathcal{A} satisfies $AB4^*$, i.e. if products are exact in \mathcal{A} , then it is trivial to observe that for any object $N \in D(\mathcal{A})$, a map

$$N \rightarrow \varprojlim_{i \geq 0} N_{\geq -i}$$

is a quasi-isomorphism. This is useful because it allows us to reduce certain questions related to unbounded complexes to bounded below complexes.

For an algebraic stack X , we will denote by $QC(X)$ the category of quasi-coherent sheaves on X . Unfortunately, $QC(X)$ rarely satisfy $AB4^*$. However, in several interesting examples, as shown by the following theorem, these abelian categories satisfy a weaker axiom, $AB4^{*-n}$, for some $n \geq 0$. This just means that the i -th derived functors of the product vanish if $i > n$ (see Section 2).

1.1. Theorem. *Let S be a separated noetherian scheme and X/S be a separated Deligne-Mumford stack of finite type. Assume*

- (1) *The coarse moduli space of X is a scheme.*
- (2) *X is covered by Zariski open substacks each of which admits a finite étale cover from a scheme.*

Then $QC(X)$ satisfies $AB4^{-n}$ for some positive integer n .*

1.2. Remark. In particular, if $S = \text{Spec}(k)$ is a field, the conditions of the above theorem are satisfied for a quotient stack X/k which is a separated Deligne-Mumford stack with quasi-projective coarse moduli space (see [Kr06], Proposition 5.2). Recall that a stack X is called a *quotient stack* if it can be expressed as a quotient of a scheme by a linear algebraic group. In the case when the stack is noetherian and normal, this is equivalent to saying that X has resolution property, i.e., every coherent sheaf can be expressed as quotient of a locally free sheaf (see [To04]).

In the following theorem, we observe that the weaker condition $AB4^{*-n}$, is still enough to make the concept of homotopy limits useful for reducing questions about unbounded complexes to bounded below complexes.

1.3. Theorem. *Let \mathcal{A} be an abelian category with enough injectives. Assume \mathcal{A} satisfies $AB4^{*-n}$ for some positive integer n . Then for every $N \in D(\mathcal{A})$, we have an isomorphism $N \rightarrow \varprojlim_{\geq -i} N_{\geq -i}$ in $D(\mathcal{A})$. In particular, the category of \mathcal{A} -complexes has enough K -injective objects. (For the definition of K -injective objects, see [Sp88])*

For an algebraic stack X we denote by $D(X)$ the unbounded derived category of quasi-coherent sheaves on X . $D(\mathcal{O}_X)$ (resp. $D^{cart}(\mathcal{O}_X)$) will stand for the unbounded derived category of (resp. cartesian) \mathcal{O}_X -modules on the lisse-étale site of X . Let $D_{qc}(\mathcal{O}_X)$ (resp. $D_{qc}^{cart}(\mathcal{O}_X)$) be the full subcategory of $D(\mathcal{O}_X)$ ($D^{cart}(\mathcal{O}_X)$) consisting of complexes with cartesian quasi-coherent cohomology. We apply Theorem (1.3) to prove the following result, which is the main result of this paper.

1.4. Theorem. *X be any algebraic stack such that the diagonal morphism $\Delta_{X/\mathbb{Z}} : X \rightarrow X \times_{\mathbb{Z}} X$ is affine. Assume that $QC(X)$ satisfies $AB4^{*-n}$ for some positive integer n . Then we have a natural equivalence*

$$D(X) \cong D_{qc}(\mathcal{O}_X) \cong D_{qc}^{cart}(\mathcal{O}_X).$$

The above theorem is already known in the case separated quasi-compact schemes (cf. [BN93]). To prove the above theorem we introduce the notion of a *strong Serre subcategory* (see Section 5). The diagonal morphism of a stack being affine will actually imply that $QC(X)$ is a strong Serre subcategory of categories of cartesian \mathcal{O}_X -modules and general \mathcal{O}_X -modules on the lisse-étale site of X . Theorem (1.4) is then the result of the following general comparison theorem.

1.5. Theorem. *Let $F : \mathcal{C} \rightarrow \mathcal{M}$ be an embedding of a strong Serre subcategory of \mathcal{M} . Assume F has a right adjoint. Let $D_{\mathcal{C}}(\mathcal{M})$ denote the full subcategory of \mathcal{M} consisting of complexes whose cohomology lies in \mathcal{C} . Assume both \mathcal{C} and \mathcal{M} satisfy AB_4^{*-n} for some nonnegative integer n . Then the natural functor*

$$RF : D(\mathcal{C}) \rightarrow D_{\mathcal{C}}(\mathcal{M})$$

is an equivalence.

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2. PRELIMINARIES

In this section we recall some well known facts about the derived functors of product and homotopy limits. For an abelian category \mathcal{A} we use the notations $K(\mathcal{A})$ and $D(\mathcal{A})$ to denote the category of complexes up to homotopy and the derived category respectively. A superscript ‘+’ (e.g. $D^+(\mathcal{A})$) will denote the respective bounded below version.

Preliminaries on Derived Functors of Product

2.1 (Derived Functors of Product). Let \mathcal{A} be an abelian category with enough injectives. Assume small products exist in \mathcal{A} . For any indexing set I , let \mathcal{A}^I be the I -fold product of \mathcal{A} . \mathcal{A}^I itself is an abelian category with enough injectives and since products exist in \mathcal{A} , we have a natural functor

$$\Pi : \mathcal{A}^I \rightarrow \mathcal{A} \quad \text{given by} \quad (A_i)_{i \in I} \rightarrow \prod_i A_i$$

This functor is additive and since it admits a right adjoint, it is left exact. We will denote by $\Pi^{(n)} A_i$ the n -th right derived functor of the above functor evaluated at $(A_i)_{i \in I}$. Concretely, in order to calculate $\Pi^{(n)} A_i$ one chooses injective resolution $0 \rightarrow A_i \rightarrow I_i^\bullet$ of A_i for each i and then $\Pi^{(n)} A_i = H^n(\Pi_i I_i^\bullet)$.

2.2. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. Assume \mathcal{A} has small products and enough injectives. Let $\{A_\alpha\}$ be a set of objects in \mathcal{A} and for each A_α and

$$0 \rightarrow A_\alpha \rightarrow C_\alpha^\bullet$$

be a resolution. Then we have the following spectral sequence

$$E_1^{pq} = \Pi^{(q)}(C^p) \Rightarrow \Pi^{(p+q)} A_\alpha$$

2.3. Definition. [Ro06] Let \mathcal{A} be an abelian category with enough injectives and in which products exist. For a nonnegative integer n , \mathcal{A} is said to satisfy the *axiom AB4*-n* if for any set I , and any collection of objects $\{A_\alpha\}_{\alpha \in I}$ of \mathcal{A}

$$\prod_{\alpha}^{(q)} A_\alpha = 0 \quad \forall q \geq N + 1$$

2.4. Remark. Let A be a noetherian local ring and let X be the complement of the closed point in $\text{Spec}(A)$. Roos showed that $QC(X)$ is AB4*- n (cf. [Ro06], Theorem 1.15), where $n = \max\{\dim(A), 1\}$, and this bound is sharp.

Preliminaries on Homotopy Limits

2.5. Definition. [BN93] Let \mathcal{T} be a triangulated category. Let

$$\cdots \xrightarrow{\alpha} M_{i+1} \xrightarrow{\alpha} M_i \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} M_2 \xrightarrow{\alpha} M_1$$

be an inverse system of objects $M_i \in \mathcal{T}$ indexed by the natural numbers \mathbb{N} . Assume $\prod M_i$ is representable in \mathcal{T} . Then the *homotopy limit* of $\{M_i\}$, denoted by $\varprojlim M_i$ is defined by the following exact triangle in \mathcal{T} .

$$\varprojlim M_i \rightarrow \prod M_i \xrightarrow{1-\alpha} \prod M_i \rightarrow \varprojlim M_i[1]$$

Dually one can also define the notion of a homotopy colimit of a sequence $M_1 \rightarrow M_2 \rightarrow \cdots$ by the following distinguished triangle.

$$\oplus M_i \xrightarrow{1-\alpha} \oplus M_i \rightarrow \varinjlim M_i \rightarrow \oplus M_i[1]$$

2.6. Remark. We recall the following features of homotopy limits from [BN93].

- (1) Since the cone of any morphism in a triangulated category is only unique up to a non-canonical isomorphism, homotopy limits are only unique up to a *non-canonical* isomorphism.
- (2) Let $\{M_i\}_{i \in \mathbb{N}}$ be an inverse system in \mathcal{T} such that $\varprojlim M_i$ exists. Let $\phi_i : L \rightarrow M_i$ be a collection of morphisms from L to M_i 's. ϕ_i is a compatible system of morphisms iff the composition $\prod \phi : L \rightarrow \prod M_i \xrightarrow{1-\alpha} \prod M_i$ is zero. Thus in this case there is an induced morphism (non-unique in general) $\phi : L \rightarrow \varprojlim M_i$. The non-uniqueness of ϕ means that in general the natural map $\text{Hom}_{\mathcal{T}}(L, \varprojlim M_i) \rightarrow \varprojlim \text{Hom}_{\mathcal{T}}(L, M_i)$ is *not* an isomorphism. In other words homotopy limit is usually not a limit. However whenever $R\text{Hom}$ makes sense, $R\text{Hom}_{\mathcal{T}}(L, \varprojlim M_i) \cong \varprojlim R\text{Hom}_{\mathcal{T}}(L, M_i)$.

2.7. Remark. Let \mathcal{A} be an abelian category with products. Let $\{M_i\}$ be a collection of complexes in $K(\mathcal{A})$ such that each M_i is K -injective. Then it is easy to see that the class of the product complex $\prod M_i$ in $D(\mathcal{A})$ represents the product of M_i 's in $D(\mathcal{A})$. In particular, if each M_i is bounded below, and \mathcal{A} has enough injectives, then $\prod M_i$ exists in $D(\mathcal{A})$. Thus for any inverse system $\{M_i\}_{i \in \mathbb{N}}$ of objects in $D(\mathcal{A})$, $\varprojlim M_i$ exists if each M_i is bounded below. One can also talk

about homotopy limits in $K(\mathcal{A})$ and one can show that the homotopy limit of K -injective objects in $K(\mathcal{A})$ is again a K -injective.

3. PRODUCTS OF QUASI-COHERENT SHEAVES ON A DELIGNE-MUMFORD STACK

Throughout this section we fix a noetherian separated base scheme S and let X/S be a separated Deligne-Mumford stack of finite type. We denote by $q : X \rightarrow \underline{X}$ the coarse moduli space of X . In particular, \underline{X} is separated.

3.1. Lemma. *Let $i : U = q^{-1}(\underline{U}) \rightarrow Y$ be an open substack such that the coarse moduli space \underline{U} of U is an affine open subset of \underline{Y} . Then the inclusion $i : U \rightarrow Y$ is affine. In particular, the functor*

$$i_* : QC(U) \rightarrow QC(Y)$$

is exact.

Proof. Left to the reader. □

3.2. Lemma. *Let Y/S be any stack and $Y = \bigcup_{i=1}^r V_i$ be a Zariski cover by finitely many open substacks. For $1 \leq k \leq r$, let*

$$U_k = \coprod_{1 \leq i_1 < i_2 < \dots < i_k \leq r} V_{i_1} \cap \dots \cap V_{i_k}$$

and $j_k : U_k \rightarrow Y$ be the natural maps. Assume the following holds.

- (i) *For every k , $j_{k*} : QC(U_k) \rightarrow QC(Y)$ is exact.*
- (ii) *For every k , $QC(U_k)$ is $AB4^*-n_k$ for some positive integer n_k .*

Then $QC(Y)$ is $AB4^-n$ for any $n \geq \max_k \{k + n_k\}$.*

Proof. Let $\{F_\alpha\}_{\alpha \in I}$ be a set of quasi-coherent sheaves on Y . For each F_α we have a Čech resolution

$$0 \rightarrow F_\alpha \rightarrow \mathcal{C}^1(F_\alpha) \rightarrow \dots \rightarrow \mathcal{C}^r(F_\alpha) \rightarrow 0$$

where for each k , $\mathcal{C}^k(F_\alpha) = j_{k*} j_k^* \mathcal{F}$.

Since j_{k*} is exact and has an exact left adjoint, it preserves products and also maps injective objects in $QC(U_k)$ to injective objects in $QC(Y)$. Since j_{k*} is exact, we have

$$j_{k*}(\Pi^{(i)} j_k^* \mathcal{F}) = (\Pi^{(i)}) \mathcal{C}^k(F_\alpha).$$

Thus from the assumption that $QC(U_k)$ is $AB4^*-n_k$, we get

$$\Pi^{(i)} \mathcal{C}^k(F_\alpha) = 0 \quad \forall i > n_k$$

Now from (2.2), we have a spectral sequence

$$E_1^{p,q} = \Pi^{(q)} \mathcal{C}^p(F_\alpha) \Rightarrow \Pi^{(p+q)} F_\alpha$$

This proves the lemma. □

3.3. Remark. If U is an affine scheme, then $QC(U)$ is $AB4^*$. Thus the above lemma immediately implies that if Y is a separated quasi-compact scheme, then $QC(Y)$ satisfies $AB4^{*-n}$, where $n + 1$ is the minimum number of affine opens required to cover X .

3.4. Lemma. *Let Y be any stack which admits a finite étale cover $f : Z \rightarrow Y$ where Z is a separated quasi-compact scheme. Then $QC(Y)$ satisfies $AB4^{*-n}$ for some positive integer n .*

Proof. Since f is finite étale, f^* is both right as well as left adjoint of f_* . Moreover both f_* , f^* are exact. Thus f^* preserves products and takes injectives to injectives. Therefore for any set of sheaves $\{\mathcal{F}_\alpha\}$ in $QC(Y)$

$$\Pi^{(i)} f^* \mathcal{F}_\alpha = f^* (\Pi^{(i)} \mathcal{F}_\alpha)$$

Since $QC(Z)$ satisfies $AB4^{*-n}$ for some positive integer n (3.3), $\Pi^{(i)} f^* \mathcal{F}_\alpha = 0 \forall i > n$. This implies $\Pi^{(i)} \mathcal{F}_\alpha = 0 \forall i > n$. \square

Proof of Theorem 1.1. The statement now is straightforward from the assumptions, Lemma (3.1) and Lemma (3.4). \square

4. HOMOTOPY LIMIT OF TRUNCATIONS

In this section we prove Theorem (1.3).

4.1. Lemma. *Let \mathcal{A} be any abelian category with products. Let $N \in D^+(\mathcal{A})$. Then for any integers i_0 and j_0*

$$\varprojlim_{i \geq i_0} N_{\geq -i} \cong \varprojlim_{i \geq j_0} N_{\geq -i}$$

Proof. Without loss of generality we may assume $j_0 = i_0 + 1$. For simplicity of notation, let $N_i = N_{\geq -i}$. We now have the following diagram where rows and last two columns are distinguished triangles.

$$\begin{array}{ccccc} 0 & \longrightarrow & N_{i_0+1} & \xrightarrow{\cong} & N_{i_0+1} \\ & & \downarrow & & \downarrow \\ \varprojlim_{i \geq i_0+1} N_i & \longrightarrow & \Pi_{i \geq i_0+1} N_i & \xrightarrow{1\text{-shift}} & \Pi_{i \geq i_0+1} N_i \\ & & \downarrow & & \downarrow \\ \varprojlim_{i \geq i_0} N_i & \longrightarrow & \Pi_{i \geq i_0} N_i & \xrightarrow{1\text{-shift}} & \Pi_{i \geq i_0} N_i \end{array}$$

By the octahedron axiom, the triangle

$$0 \rightarrow \varprojlim_{i \geq i_0+1} N_i \rightarrow \varprojlim_{i \geq i_0} N_i$$

must be distinguished. This proves the lemma. \square

4.2. Lemma. *Let \mathcal{A} be any abelian category with products and A be an object of \mathcal{A} . Then the sequence*

$$A \rightarrow \prod_{i \geq 0} A \xrightarrow{1\text{-shift}} \prod_{i \geq 0} A$$

is exact.

Proof. Left to the reader. \square

4.3. Lemma. *Let \mathcal{A} be any abelian category and $N \in D^+(\mathcal{A})$. Then for any integer i_0 ,*

$$N \cong \varprojlim_{i \geq i_0} N_{\geq -i}$$

Proof. By Lemma (4.1) we may assume

$$H^{-i}(N) = 0 \quad \forall i \geq i_0$$

In this case choose a complex C^\bullet representing N such that $C^{-i} = 0 \quad \forall i \geq i_0$. Thus $C_{\geq -i}^\bullet = C^\bullet \quad \forall i \geq i_0$. The lemma now follows because by Lemma (4.2) the following is an exact sequence of complexes

$$0 \rightarrow C^\bullet \rightarrow \prod_{i \geq i_0} C_{\geq -i}^\bullet \xrightarrow{1\text{-shift}} \prod_i C_{\geq -i}^\bullet \rightarrow 0$$

\square

4.4. Lemma. *Let \mathcal{A} be an abelian category which is AB_4^{*-n} . Let $\{M_\alpha\}$ and $\{L_\alpha\}$ be a collection of bounded below complexes of injective objects. Let t be an integer and let $\phi_\alpha : M_\alpha \rightarrow L_\alpha$ be a collection of morphisms such that the induced map*

$$H^i(M_\alpha) \rightarrow H^i(L_\alpha)$$

is an isomorphism for all $i \geq t$. Then for each $i \geq t + n + 1$,

$$H^i(\prod M_\alpha) \rightarrow H^i(\prod L_\alpha)$$

is an isomorphism.

Proof. For a collection of bounded below complexes $\{N_\alpha\}$ of injective objects, we have the following spectral sequence

$$E_2^{p,q} = \prod_\alpha^{(p)} H^q(N_\alpha) \Rightarrow H^{p+q}(\prod_\alpha N_\alpha).$$

Since $E_2^{p,q} = 0$ for $p < 0$ or $p > n$, the spectral sequence is convergent. Now we apply it two both $\{M_\alpha\}$ and $\{L_\alpha\}$. \square

Proof of (1.3). Let $N \in Kom(\mathcal{A})$ be a complex. Fix an integer t . Let $N_{\geq -t}$ be the truncation complex and L its injective resolution with $\tau : N \rightarrow L$ the induced map. We denote by M_i an injective resolution of $N_{\geq -i}$ and let

$$\tilde{\tau} : \prod_i M_i \rightarrow \prod_i L_{\geq -i}$$

be a map induced by τ . We have the following commutative diagram.

$$\begin{array}{ccccc}
\varprojlim M_i & \longrightarrow & \Pi_i M_i & \xrightarrow{1-shift} & \Pi_i M_i \\
\downarrow \tau & & \downarrow \tilde{\tau} & & \downarrow \tilde{\tau} \\
L & \xrightarrow{h_L} & \Pi_i L_{\geq -i} & \xrightarrow{1-shift} & \Pi_i L_{\geq -i}
\end{array}$$

Note that the top row defines a distinguished triangle by definition of $\varprojlim M_i$ and the bottom row is an exact sequence by Lemma (4.3) and hence also defines a distinguished triangle. We also have maps $h_N : N \rightarrow \varprojlim M_i$ and $\tau : N \rightarrow L$. Thus it gives a following diagram for each k .

$$\begin{array}{ccccccc}
H^k(N) & & & & & & \\
& \searrow & & & & & \\
& & H^k(\varprojlim M_i) & \xrightarrow{h_M} & H^k(\Pi_i M_i) & \xrightarrow{1-shift} & H^k(\Pi_i M_i) \\
& & \downarrow \tau & & \downarrow \tilde{\tau} & & \downarrow \tilde{\tau} \\
& & H^k(L) & \xrightarrow{h_L} & H^k(\Pi_i L_{\geq -i}) & \xrightarrow{1-shift} & H^k(\Pi_i L_{\geq -i})
\end{array}$$

It follows from Lemma (4.4) that the map $H^k(\Pi_i M_i) \rightarrow H^k(\Pi_i L_{\geq -i})$ is an isomorphism for $k \geq n - t + 1$. Therefore, $H^k(\varprojlim M_i) \cong H^k(L)$ for $k \geq n - t + 2$. Since $H^k(N) \rightarrow H^k(L)$ is an isomorphism for $k \geq -t$, we see that for all k large enough as compared to $-t$, $H^k(N) \rightarrow H^k(\varprojlim M_i)$ is an isomorphism. But t was arbitrary. Therefore $N \rightarrow \varprojlim M_i$ is a quasi-isomorphism. \square

5. STRONG SERRE SUBCATEGORIES

Let \mathcal{M} denote an abelian category and \mathcal{C} denote a full subcategory of \mathcal{M} . Recall the following definition of a Serre category.

5.1. Definition (Serre subcategory). \mathcal{C} is called a *Serre subcategory* if for any objects a, b of \mathcal{C} , and any exact sequence

$$0 \rightarrow a \rightarrow c \rightarrow b \rightarrow 0$$

c is also an object of \mathcal{C} .

Throughout this section we make the following assumptions on \mathcal{M} and \mathcal{C}

- \mathcal{M} and \mathcal{C} have enough injectives.
- The inclusion functor $F : \mathcal{C} \rightarrow \mathcal{M}$ is exact.

In this case, saying \mathcal{C} a Serre subcategory is equivalent to saying that for any objects a, b of \mathcal{C} , the natural map

$$Ext_{\mathcal{C}}^1(a, b) \rightarrow Ext_{\mathcal{M}}^1(a, b)$$

is an isomorphism. This as well as examples in algebraic geometry which will be mentioned below motivate the following definition.

5.2. Definition. \mathcal{C} is called a *strong Serre subcategory* if for any two objects a, b of \mathcal{C} , the natural maps

$$\mathrm{Ext}_{\mathcal{C}}^i(a, b) \rightarrow \mathrm{Ext}_{\mathcal{M}}^i(a, b)$$

are isomorphisms for all $i \geq 0$.

The motivation for the definition comes from the following easy proposition.

5.3. Proposition. *Let \mathcal{C} be a full subcategory of \mathcal{M} . Then \mathcal{C} is a strong Serre subcategory of \mathcal{M} iff the natural functor*

$$D^+(\mathcal{C}) \rightarrow D_{\mathcal{C}}^+(\mathcal{M})$$

is fully faithful.

We recall the following theorem (not stated in its full generality) about existence of adjoints which essentially follows from Freyd adjoint theorem.

5.4. Theorem (Freyd Adjoint Functor theorem). *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be any functor of abelian categories. Assume \mathcal{A} is cocomplete, i.e. arbitrary small colimits exists in \mathcal{A} and that \mathcal{A} has a generator. Then F has a right adjoint iff it preserves colimits.*

5.5. Remark. In the situations of our interest the functor F above will usually be a fully faithful embedding of the category \mathcal{C} of quasi-coherent sheaves on an Artin stack in a bigger category \mathcal{M} , where \mathcal{M} will either be the category of cartesian \mathcal{O}_X -modules on the lisse-étale site of X or the category of all \mathcal{O}_X -modules on the lisse-étale site of X (cf. [LM00], [Ol07]). In these cases the functor $\mathcal{C} \rightarrow \mathcal{M}$ is a fully faithful embedding which preserves colimits. Moreover \mathcal{C} is a cocomplete category. The existence of a generator in \mathcal{C} can be proved using the fact that any quasi-coherent sheaf is a colimit of coherent sheaves together with the observation that isomorphism classes of coherent sheaves on a stack form a ‘set’. Thus, in this situation $\mathcal{C} \rightarrow \mathcal{M}$ will have a right adjoint. Such embeddings $\mathcal{C} \rightarrow \mathcal{M}$ will also be the main examples of strong Serre subcategories in this paper. For more background, see [LM00] and [Ol07].

5.6. Theorem. *Let X/\mathbb{Z} be an algebraic stack such that the diagonal*

$$\Delta_{X/\mathbb{Z}} : X \rightarrow X \times_{\mathbb{Z}} X$$

is affine. Let $\mathcal{C} = QC(X)$, $\mathrm{Mod}^{\mathrm{cart}}(X)$ and $\mathrm{Mod}(X)$ be the category of all cartesian \mathcal{O}_X -modules and all \mathcal{O}_X -modules, respectively, on the lisse-étale site of X . Then \mathcal{C} is a strong Serre subcategory of $\mathrm{Mod}^{\mathrm{cart}}(X)$ and $\mathrm{Mod}(X)$.

Proof. We first prove the case when X is a affine scheme. For $\mathrm{Mod}^{\mathrm{cart}}(X)$, it essentially follows from the fact that every injective quasicoherent sheaf on X is flabby (cf. [Gr57]); for $\mathrm{Mod}(X)$, we observe that the inclusion of $\mathrm{Mod}^{\mathrm{cart}}(X)$ in $\mathrm{Mod}(X)$ has a exact right adjoint. The remaining part of the proof for $\mathrm{Mod}^{\mathrm{cart}}(X)$ and $\mathrm{Mod}(X)$ is the same, thus we will only argue for $\mathrm{Mod}^{\mathrm{cart}}(X)$. Now for the general case, let $Y \rightarrow X$ be any fppf cover where Y is an affine

scheme. We let Y_\bullet denote the simplicial scheme which is the 0-coskeleton of this cover. Concretely

$$Y_i = Y \times_X Y \times_X \cdots \times_X Y \quad (i\text{-times})$$

Let $f_i : Y_i \rightarrow X$ denote the natural morphism. Note that by assumptions on X , each Y_i is an affine scheme and moreover the morphisms f_i are also affine. We denote by f_{i*} the functor $QC(Y_i) \rightarrow \mathcal{M}od^{cart}(X)$ and f'_{i*} the functor from $\mathcal{M}od^{cart}(Y_i) \rightarrow \mathcal{M}od^{cart}(X)$. We claim that for any quasi-coherent sheaf \mathcal{G} on Y_i ,

$$(1) \quad R^k f_{i*}(\mathcal{G}) = R^k f'_{i*}(\mathcal{G}) = 0 \quad \forall i > 0$$

The vanishing of $R^k f_{i*}(\mathcal{G})$ for $i > 0$ follows from the fact that f_i is affine. The equality

$$R^k f_{i*}(\mathcal{G}) = R^k f'_{i*}(\mathcal{G})$$

follows from the fact that any injective sheaf on Y_i is flabby.

For any $\mathcal{F} \in QC(X)$, there is a simplicial resolution

$$\mathcal{F} \rightarrow f_{1*} f_1^*(\mathcal{F}) \rightarrow f_{2*} f_2^*(\mathcal{F}) \rightarrow \cdots,$$

Moreover, for any $\mathcal{G} \in \mathcal{C}$, we have the following two spectral sequences

$$Ext_X^p(\mathcal{G}, f_{q*} f_q^* \mathcal{F}) \Rightarrow Ext_X^{p+q}(\mathcal{G}, \mathcal{F})$$

$$\widetilde{Ext}_X^p(\mathcal{G}, f_{q*} f_q^* \mathcal{F}) \Rightarrow \widetilde{Ext}_X^{p+q}(\mathcal{G}, \mathcal{F})$$

where henceforth for simplicity we write

$$Ext_X^p(-, -) = Ext_{QC(X)}^p(-, -)$$

$$\widetilde{Ext}_X^p(-, -) = Ext_{\mathcal{M}od^{cart}(X)}^p(-, -)$$

There is a natural morphism from the first spectral sequence to the second and to prove the theorem it is enough to show that for any q

$$Ext_X^p(\mathcal{G}, f_{q*} f_q^* \mathcal{F}) \rightarrow \widetilde{Ext}_X^p(\mathcal{G}, f_{q*} f_q^* \mathcal{F})$$

is an isomorphism. But by (1), we have natural isomorphisms

$$Ext_X^p(\mathcal{G}, f_{q*} f_q^* \mathcal{F}) \cong Ext_{Y_q}(f_q^* \mathcal{G}, f_q^* \mathcal{F})$$

$$\widetilde{Ext}_X^p(\mathcal{G}, f_{q*} f_q^* \mathcal{F}) \cong \widetilde{Ext}_{Y_q}(f_q^* \mathcal{G}, f_q^* \mathcal{F})$$

But Y_q is an affine scheme. Hence we already know that $QC(Y_q)$ is a strong Serre subcategory of $\mathcal{M}od^{cart}(Y)$, in particular that

$$Ext_{Y_q}(f_q^* \mathcal{G}, f_q^* \mathcal{F}) = \widetilde{Ext}_{Y_q}(f_q^* \mathcal{G}, f_q^* \mathcal{F}).$$

This proves the result. \square

Proof of Theorem (1.5). We denote by $G : \mathcal{M} \rightarrow \mathcal{C}$ the right adjoint of F . By Proposition (5.3) we already know that

$$RF^+ : D^+(\mathcal{C}) \rightarrow D_{\mathcal{C}}^+(\mathcal{M})$$

is fully faithful. The assumption that \mathcal{C} has enough injective objects implies RF^+ is essential surjective and hence an equivalence. The inverse of RF^+ is given by the restriction of RG^+ to $D_{\mathcal{C}}^+(\mathcal{M})$. We now need to prove that

$$RF : D(\mathcal{C}) \rightarrow D_{\mathcal{C}}(\mathcal{M})$$

is an equivalence for which we will apply (1.3).

Step 1: For any object $A \in D(\mathcal{C})$, $F(A) = 0$ implies $A = 0$. This implies that $\overline{F} : \overline{D}(\mathcal{C}) \rightarrow D_{\mathcal{C}}(\mathcal{M})$ is full and essentially surjective we need to show that the adjugant $F \circ RG \rightarrow Id$ is an equivalence. Let $D \in D_{\mathcal{C}}(\mathcal{M})$. We will show that the natural map $F(RG(D)) \rightarrow D$ is an isomorphism. We fix the following notation.

- I^\bullet a K -injective complex representing D .
- $C = RG(D)$. Thus $C^\bullet = G(I^\bullet)$ represents C .
- I_j^\bullet an injective bounded below complex representing $D_{\geq i}$.
- For any complex A^\bullet , ${}^s A_{\geq i}^\bullet$ denotes the i -th stupid truncation.
- For a complex A^\bullet , the same letter without the dot A , will denote the class in the derived category.
- For a fixed i , we have the inverse system of complexes $\{{}^s G(I_j^\bullet)_{\geq i}\}_j$. We define M_i^\bullet to be an object such that the following triangle is a distinguished triangle.

$$M_i^\bullet \rightarrow \Pi_j^{\mathcal{C}} {}^s G(I_j^\bullet)_{\geq i} \xrightarrow{1-\text{shift}} \Pi_j^{\mathcal{C}} G(I_j^\bullet)_{\geq i}$$

Step 2: We compare M_i and C . We claim that there is a map $M_i \rightarrow C$ such that $H^k(M_i) \rightarrow H^k(C)$ is an isomorphism for all $k > i$. This follows from the fact that stupid truncation commutes with product and we have the following diagram of distinguished triangles.

$$\begin{array}{ccccc} M_i & \longrightarrow & {}^s(\Pi_j^{\mathcal{C}} G(I_j))_{\geq i} & \xrightarrow{1-\text{shift}} & {}^s(\Pi_j^{\mathcal{C}} G(I_j))_{\geq i} \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & \Pi_j^{\mathcal{C}} G(I_j) & \xrightarrow{1-\text{shift}} & \Pi_j^{\mathcal{C}} G(I_j) \end{array}$$

The bottom triangle is a distinguished triangle since it is obtained by applying RG to the following distinguished triangle

$$D \rightarrow \Pi_j I_j \xrightarrow{1-\text{shift}} \Pi_j I_j$$

which is from Theorem (1.3) and the assumption that \mathcal{M} is AB4*-n.

Step 3: We now compare M_i^\bullet and $G(I_i^\bullet)$. We claim that there is a map $M_i \rightarrow \overline{G(I_i)}$ which induces an isomorphism

$$H^k(M_i) \rightarrow H^k(G(I_i)) \quad \forall k > i + n + 1$$

Clearly for $j \leq i$, ${}^sG(I_j^\bullet)_{\geq i} \rightarrow G(I_i)$ induces an isomorphism on the k -th cohomology if $k \geq i + 1$. After taking prodcut over j , we get a map

$$\Pi_j^{\mathcal{C}}({}^sG(I_j^\bullet)_{\geq i}) \rightarrow \Pi_j^{\mathcal{C}}I_i^\bullet$$

Since \mathcal{C} is AB4^*-n , the above map induces an isomorphism

$$H^k(\Pi_j^{\mathcal{C}}{}^sG(I_j^\bullet)_{\geq i}) \rightarrow H^k(\Pi_j^{\mathcal{C}}I_i^\bullet) \quad k \geq i + n + 1$$

The claim now follows from the following diagram of distinguished triangles.

$$\begin{array}{ccccc} M_i & \longrightarrow & \Pi_{j \leq i}^{\mathcal{C}} {}^sG(I_j)_{\geq i} & \xrightarrow{1\text{-shift}} & \Pi_{j \leq i}^{\mathcal{C}} {}^sG(I_j)_{\geq i} \\ \downarrow & & \downarrow & & \downarrow \\ G(I_i) & \longrightarrow & \Pi_j^{\mathcal{C}} G(I_i) & \xrightarrow{1\text{-shift}} & \Pi_j^{\mathcal{C}} G(I_i) \end{array}$$

Step 4: Now consider the following commutative diagram

$$\begin{array}{ccccc} F(M_i) & \longrightarrow & F(C) & \longrightarrow & D \\ & \searrow & \downarrow & \swarrow & \downarrow \\ & & F \circ G(I_i) & \longrightarrow & I_i \end{array}$$

From Steps 2 and 3, and since F is exact, it follows that

$$H^k(F(C)) \rightarrow H^k(F \circ G(I_i))$$

is an isomorphism for $k > i + n + 1$. Moreover, since I_i is a bounded below complex, by Proposition (5.3), $F \circ G(I_i) \rightarrow I_i$ is a quasi-isomorphism. Also, by definition of I_i , $H^k(D) \rightarrow H^k(I_i)$ is an isomorphism for $k \geq i$. Hence we conclude that

$$H^k(F(C)) \rightarrow H^k(D)$$

is an isomorphism for all $k > i + n + 1$. Since i is arbitrary, this proves the theorem. \square

Finally we give an example of a subcategory which is not a strong Serre subcategory. The following example also shows that the hypothesis that $\Delta_{X/\mathbb{Z}}$ is affine in Theorem (1.5) cannot be dropped.

5.7. Example. Let k be a field and let A/k be any abelian variety of positive dimension. In particular $H^1(A, \mathcal{O}_X) \neq 0$. We let $X = [\text{Spec}(k)/A]$ and \mathcal{C} denote the category of quasi-coherent sheaves on X . Let \mathcal{M} denote the category of all \mathcal{O}_X -modules on the lisse-étale site of X . \mathcal{C} is a Serre subcategory of \mathcal{M} . We claim that it is not a strong Serre subcategory.

To see this, let $\pi : \mathrm{Spec}(k) \rightarrow X$ denote the natural projection. The functor π_* from lisse-étale sheaves on $\mathrm{Spec}(k)$ to those on X is smooth and hence has an exact left adjoint. Therefore π_* maps injectives to injectives and thus we have a Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q \pi_* \mathcal{O}_X) \implies H^{p+q}(\mathrm{Spec}(k), \mathcal{O}_X)$$

which, together with the vanishing of $H^1(\mathrm{Spec}(k), \mathcal{O}_X)$ gives us an injection

$$0 \rightarrow H^0(X, R^1 \pi_* \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X)$$

But one can show that $H^0(X, R^1 \pi_* \mathcal{O}_X) = H^1(A, \mathcal{O}_X) \neq 0$. Thus $\mathrm{Ext}_{\mathcal{M}}^2(\mathcal{O}_X, \mathcal{O}_X) \neq 0$. However, \mathcal{C} itself is a semisimple category and hence $\mathrm{Ext}_{\mathcal{C}}^2(\mathcal{O}_X, \mathcal{O}_X) = 0$.

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